A bridge between Schrödinger equation and Schrödinger Bridge process *by Léon Brenig and Marc Vincke.* Université Libre de Bruxelles.

Summary *: The unitary evolution described by the Schrödinger equation and the non-unitary evolution governed by the Schrödinger Bridge random process are shown to be mathematically related. Indeed, these two types of evolution mix under nonlinear gauge transformations of the wavefunctions introduced in this work. After such a transformation the new wavefunction appears to obey again to both the above unitary and non-unitary evolutions. The interpretation of this result is discussed but remains an open question.*

Nonlinear gauge transformations of the wavefunction are introduced in the non-relativistic description of a free spinless particle of mass m. These transformations constitute a one-parameter Lie group. For wavefunctions $\psi(x) = \rho(x)^{1/2} e^{\frac{i}{\hbar} s(x)}$ normed to one, they only act on the argument $s : s \to s(\alpha) = e^{-\alpha} s$; $\alpha \in \mathbb{R}$. These transformations conserve the product $\psi\psi^*$ and keep invariant the following system of equations for the wavefunction :

$$i\partial_t \psi = -\frac{\hbar}{2m} \nabla^2 \psi \tag{0.1}$$

$$i\partial_\tau \psi = -\frac{\hbar}{2m} \nabla^2 \psi + \frac{\hbar}{m} \psi \frac{\nabla^2 \mid \psi \mid}{\mid \psi \mid} \tag{0.2}$$

The first equation is the Schrödinger equation which describes the unitary evolution in time t. The second equation is nonlinear and describes the evolution of ψ in a parameter τ that has temporal physical dimension but, a priori, is different from time t. The invariance of the system (0.1), (0.2) under the nonlinear gauge transformations is ensured provided the couple (t, τ) transforms in a hyperbolic rotation :

$$t(\alpha) = \cosh(\alpha) t + \sinh(\alpha) \tau \tag{0.3}$$

$$\tau(\alpha) = \sinh(\alpha) t + \cosh(\alpha) \tau \tag{0.4}$$

Remarkably, the equation (0.2) can exactly be transformed into the so-called Schrödinger Bridge random process that E.Schrödinger studied in 1931-1932 [1][2] :

$$\partial_\tau \varphi = \frac{\hbar}{2m} \nabla^2 \varphi \tag{0.5}$$

$$\partial_\tau \phi = -\frac{\hbar}{2m} \nabla^2 \phi \tag{0.6}$$

These two equations, respectively, describe the forward and backward diffusive evolution of a Brownian particle whose probability density is prescribed both at the initial and final time. The functions of position and time φ and ϕ are not themselves probabilities but their product $\varphi\phi$ is the probability density that interpolates at intermediate times between the initial and final distributions. In his 1931 and 1932 papers, E.Schrödinger introduced this problem in an effort to understand the origin of the Born rule in quantum mechanics. He, however, called his attempt a fiasco because he did not find any connection between quantum mechanics and the classical random process he had defined and studied. Here, we show that this is not the case : the two types of evolutions are tightly bound by the nonlinear gauge transformations defined above. We also show that this mathematical relation can be extended to interacting particles and, even, to quantum fields. A question that remains open is its physical interpretation. Some arguments could relate the Schrödinger Bridge evolution to the reduction of the wavefunction induced by a measurement. However, this hypothesis remains to be substantiated.

Références

[1] Schrödinger E. Une analogie entre la mécanique ondulatoire et quelques problèmes de probabilités en physique classique. See Chapitre VII p.296 in E. Schrödinger. Ann. Inst. Henri Poincaré, Vol. 2 (1932), p. 269-310.

[2] Schrödinger E. On the reversal of time in natural law. English translation by D.H.Delphenich of « Über die Umkehrung der Naturgesetze ». Sitz. preuss. Akad. Wiss., Phys.-Math. Klasse 9 (1931), 3-12.